JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **111**, 114–118 (1985)

Nonlinear Plasma Response

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The decomposition method (G. Adomian, "Stochastic Systems," Academic Press, New York/London, 1983; G. Adomian, "Stochastic Systems II," in press) (and earlier work of Adomian referenced therein) allows solution without linearization of problems normally requiring the self-consistent field technique. The latter technique can provide a sufficiently accurate approximation in weakly coupled plasma, but the decomposition method of Adomian provides a superior and physically preferable methodology for the general case. © 1985 Academic Press, Inc.

The response of an interacting electron stream to electrostatic perturbation is widely considered to be best calculated by the self-consistent-field technique [1] developed by Ehrenreich and Cohen [2] and Goldstone and Gottfried [3]. In seeking the equation for the charge density, use of the self-consistent field method requires taking the expectation value of the equation of motion whereas one should solve the equation first as in the decomposition method [4, 5] and then average. The usual method is a typical hierarchy procedure which necessarily neglects interaction terms simply replacing an average of a product by a product of averages. While this can be a sufficiently accurate approximation in weakly coupled plasmas, the error in doing so in general has been well established by Adomian [4, 6], Adomian and Malakian [7]. It is in fact (as shown in [4]) a perturbation method in which the instantaneous field experienced by an electron does not deviate much from the average value. We show here using the methodology of [4, 5] that a better, and certainly more physically realistic method of solution is possible now.

The total charge density consists of two parts—an external charge density ρ_{ext} and an induced charge density ρ_{ind} due to the nonuniformity of the plasma. The electrostatic potential in a single-component plasma can be calculated from Poisson's equation

$$\nabla^2\phi = -\rho_{\mathrm{ext}} + \alpha(e^{-\beta\phi} - 1) \tag{1}$$

where the second term on the right is the induced charge ρ_{ind}. Thus, we have a nonlinear differential equation for ϕ. At this point it is customary to

0022-247X/85 $3.00

make various approximations, e.g., ρ_{ext} is assumed small so that the induced potential will also be small ($\beta\phi \ll 1$) and one can write a linearized equation. However, we make no linearizing assumption. We write (1) in the standard form of [4, 5] as

$$\nabla^2\phi - \alpha e^{-\beta\phi} = -\rho_{\text{ext}} - \alpha \tag{2}$$

or $F\phi \equiv L\phi + N\phi = g$, where $L = \nabla^2$ and $N\phi = -\alpha e^{-\beta\phi}$ is a nonlinear term, and $g = -\rho_{\text{ext}} - \alpha$. Let $L = L_x + L_y + L_z$, where $L_x = \partial^2/\partial x^2$, $L_y = \partial^2/\partial y^2$, $L_z = \partial^2/\partial z^2$ (see [5, 9]). Then

$$[L_x + L_y + L_z]\,\phi + N\phi = g.$$

We solve for $L_x\phi$, $L_y\phi$, $L_z\phi$ in turn as discussed in [5, 8] so that

$$L_x\phi = g - L_y\phi - L_z\phi - N\phi$$
$$L_y\phi = g - L_z\phi - L_x\phi - N\phi$$
$$L_z\phi = g - L_x\phi - L_y\phi - N\phi.$$

The inverses L_x^{-1}, L_y^{-1}, L_z^{-1} are defined [5, 8] as two-fold definite integrations (thus L_x is a double integration between the limits 0 to x, etc.) If we indicate the homogeneous solutions by Φ_x, Φ_y, Φ_z (i.e., $L_x\Phi_x = 0$, etc.)

$$\phi = \Phi_x + L_x^{-1}g - L_x^{-1}L_y\phi - L_x^{-1}L_z\phi - L_x^{-1}N\phi$$
$$\phi = \Phi_y + L_y^{-1}g - L_y^{-1}L_z\phi - L_y^{-1}L_x\phi - L_y^{-1}N\phi$$
$$\phi = \Phi_z + L_z^{-1}g - L_z^{-1}L_x\phi - L_z^{-1}L_y\phi - L_z^{-1}N\phi.$$

Adding (and dividing by 3),

$$\begin{aligned}\phi = {} & (\tfrac{1}{3})\{(\Phi_x + \Phi_y + \Phi_z) + (L_x^{-1} + L_y^{-1} + L_z^{-1})\,g\} \\ & - (\tfrac{1}{3})\{L_x^{-1}L_y + L_y^{-1}L_z + L_z^{-1}L_x + L_x^{-1}L_z + L_y^{-1}L_x + L_z^{-1}L_y\}\,\phi \\ & - \tfrac{1}{3}\{L_x^{-1} + L_y^{-1} + L_z^{-1}\}\,N\phi.\end{aligned}$$

We define the entire first of the above three lines as ϕ_0 in the decomposition $\phi = \sum_{n=0}^{\infty}\phi_n$. Define the entire second line as $-(L^{-1}R)\,\phi$ and the third line as $L^{-1}\,N\phi$ then we have the convenient form

$$\phi = \phi_0 - (L^{-1}R)\,\phi - L^{-1}(N\phi). \tag{3}$$

For ϕ we substitute $\phi = \sum_{n=0}^{\infty}\phi_n$ and $N\phi$ is developed in Adomian's A_n polynomials [4, 5] thus

$$\sum_{n=0}^{\infty}\phi_n = \phi_0 - (L^{-1}R)\sum_{n=0}^{\infty}\phi_n - L^{-1}\sum_{n=0}^{\infty}A_n. \tag{4}$$

The A_n of course are developed for the particular nonlinearity present [4, 5, 8]. The components of ϕ are

$$\begin{aligned}
\phi_0 &= (\tfrac{1}{3})\{(\Phi_x+\Phi_y+\Phi_z)+(L_x^{-1}+L_y^{-1}+L_z^{-1})\,g\}\\
\phi_1 &= -L^{-1}R\phi_0-L^{-1}A_0\\
\phi_2 &= -L^{-1}R\phi_1-L^{-1}A_1\\
\phi_3 &= -L^{-1}R\phi_2-L^{-1}A_2\\
&\vdots
\end{aligned} \tag{5}$$

The A_n have been previously defined (see [4, 5, 8, 9]) for $N\phi=\alpha e^{-\beta\phi}$ they are[1]

$$\begin{aligned}
A_0 &= \alpha e^{-\beta\phi_0}\\
A_1 &= \alpha e^{-\beta\phi_0}[-\beta\phi_1]\\
A_2 &= \alpha e^{-\beta\phi_0}[-\beta\phi_2+(\beta^2/2!)\,\phi_1^2]\\
A_3 &= \alpha e^{-\beta\phi_0}[-\beta\phi_3+\beta^2\phi_1\phi_2-(\beta^3/3!)\,\phi_1^3]\\
A_4 &= \alpha e^{-\beta\phi_0}[-\beta\phi_4+\beta^2\{(\phi_2^2/2!)+\phi_1\phi_3\}\\
&\quad -(\beta^3\phi_1^2\phi_2/2!)+(\beta^4\phi_1^4/4!)]\\
A_5 &= \alpha e^{-\beta\phi_0}[-\beta\phi_5+\beta^2\{\phi_2\phi_3+\phi_1\phi_4\}\\
&\quad -\beta^3\{(\phi_1\phi_2^2/2!)+(\phi_1^2\phi_3/2!)\}\\
&\quad +(\beta^4\phi_1^3\phi_2/3!)-(\beta^5\phi_1^5/5!)]\\
A_6 &= \alpha e^{-\beta\phi_0}[-\beta\phi_6+\beta^2\{(\phi_3^2/2!)+\phi_2\phi_4+\phi_1\phi_5\}\\
&\quad -\beta^3\{(\phi_2^3/3!)+\phi_1\phi_2\phi_3+\phi_1^2\phi_4/2!\}\\
&\quad +\beta^4\{(\phi_1^2/2!)(\phi_2^2/2!)+(\phi_1^3/3!)\,\phi_3\}\\
&\quad -\beta^5(\phi_1^4/4!)\,\phi_2+\beta^6(\phi_1^6/6!)]\\
A_7 &= \alpha e^{-\beta\phi_0}[-\beta\phi_7+\beta^2\{\phi_3\phi_4+\phi_2\phi_5+\phi_1\phi_6\}\\
&\quad -\beta^3\{(\phi_2^2/2!)\,\phi_3+\phi_1(\phi_3^2/2!)+\phi_1\phi_2\phi_4\\
&\quad +(\phi_1^2/2!)\,\phi_5\}+\beta^4\{\phi_1(\phi_2^3/3!)+(\phi_1^2/2!)\,\phi_2\phi_3\\
&\quad +(\phi_1^3/3!)\,\phi_4\}-\beta^5\{(\phi_1^3/3!)(\phi_2^2/2!)+(\phi_1^4/4!)\,\phi_3\}\\
&\quad +\beta^6(\phi_1^5/5!)\,\phi_2-\beta^7(\phi_1^7/7!)]
\end{aligned} \tag{6}$$

[1] We can let $N\phi=e^{-\beta\phi}$ and write αNy instead of Ny.

$$\begin{aligned}
A_8 = \alpha e^{-\beta\phi_0}[& -\beta\phi_8 + \beta^2\{(\phi_4^2/2!) + \phi_3\phi_5 + \phi_2\phi_6 + \phi_1\phi_7\} \\
& -\beta^3\{\phi_2(\phi_3^2/2!) + (\phi_2^2/2!)\,\phi_4 + \phi_1\phi_3\phi_4 \\
& \quad + \phi_1\phi_2\phi_5 + (\phi_1^2/2!)\,\phi_6\} \\
& + \beta^4\{(\phi_2^4/4!) + \phi_1(\phi_2^2/2!)\,\phi_3 + (\phi_1^2/2!)(\phi_3^2/2!) \\
& \quad + (\phi_1^2/2!)\,\phi_2\phi_4 + (\phi_1^3/3!)\,\phi_5\} \\
& - \beta^5\{(\phi_1^2/2!)(\phi_2^3/3!) + (\phi_1^3/3!)\,\phi_2\phi_3 + (\phi_1^4/4!)\,\phi_4\} \\
& + \beta^6\{(\phi_1^4/4!)(\phi_2^2/2!) + (\phi_1^5/5!)\,\phi_3\} \\
& \quad - \beta^7(\phi_1^6/6!)\,\phi_2 + \beta^8(\phi_1^8/8!)] \\
A_9 = \alpha e^{-\beta\phi_0}[& -\beta\phi_9 + \beta^2\{\phi_4\phi_5 + \phi_3\phi_6 + \phi_2\phi_9 + \phi_1\phi_8\} \\
& - \beta^3\{(\phi_3^3/3!) + \phi_2\phi_3\phi_4 + (\phi_2^2/2!)\,\phi_5 + \phi_1(\phi_4^2/2!) \\
& \quad + \phi_1\phi_3\phi_5 + \phi_1\phi_2\phi_6 + (\phi_1^2/2!)\,\phi_7\} \\
& + \beta^4\{(\phi_2^3/3!)\,\phi_3 + \phi_1\phi_2(\phi_3^2/2!) + \phi_1(\phi_2^2/2!)\,\phi_4 \\
& \quad + (\phi_1^2/2!)\,\phi_3\phi_4 + (\phi_1^2/2!)\,\phi_2\phi_5 + (\phi_1^3/3!)\,\phi_6\} \\
& - \beta^5\{\phi_1(\phi_2^4/4!) + (\phi_1^2/2!)(\phi_2^2/2!)\,\phi_3 \\
& \quad + (\phi_1^3/3!)(\phi_3^2/2!) + (\phi_1^3/3!)\,\phi_2\phi_4 + (\phi_1^4/4!)\,\phi_5\} \\
& + \beta^6\{(\phi_1^3/3!)(\phi_2^3/3!) + (\phi_1^4/4!)\,\phi_2\phi_3 + (\phi_1^5/5!)\,\phi_4\} \\
& - \beta^7\{(\phi_1^5/5!)(\phi_2^2/2!) + (\phi_1^6/6!)\,\phi_3\} \\
& \quad + \beta^8(\phi_1^7/7!)\,\phi_2 - \beta^9(\phi_1^9/9!)]
\end{aligned}$$

which is sufficient to give us the 10-term approximation ϕ_{10}. (More are easily calculated if the problem warrants it, but we expect a very rapidly damped oscillating convergence from results discussed previously—particularly in [5].)

Since $\beta\phi$ is generally assumed small ($\beta\phi \ll 1$), $e^{-\beta\phi} - 1$ becomes $-\beta\phi$ to make a linear equation. The decomposition method solves Eq. (1) *without linearizing assumptions, perturbative methods, or truncations* and the complete solution is given by

$$\phi_n = \sum_{\nu=0}^{n-1} \phi_\nu$$

with the ϕ_ν given by (5) using the A_n in (6).

Complete and accurate solution can now be obtained for any specific initial/boundary conditions for (1) and values of α, β, ρ_{ext} in a rapidly convergent series [10] to any desired approximation.

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